

# Metric structures in $L_1$ : Dimension, snowflakes, and average distortion

James R. Lee\*  
U.C. Berkeley  
jrl@cs.berkeley.edu

Manor Mendel†  
University of Illinois  
mendelma@gmail.com

Assaf Naor  
Microsoft Research  
anaor@microsoft.com

## Abstract

We study the metric properties of finite subsets of  $L_1$ . The analysis of such metrics is central to a number of important algorithmic problems involving the cut structure of weighted graphs, including the Sparsest Cut Problem, one of the most compelling open problems in the field of approximation algorithms. Additionally, many open questions in geometric non-linear functional analysis involve the properties of finite subsets of  $L_1$ .

We present some new observations concerning the relation of  $L_1$  to dimension, topology, and Euclidean distortion. We show that every  $n$ -point subset of  $L_1$  embeds into  $L_2$  with average distortion  $O(\sqrt{\log n})$ , yielding the first evidence that the conjectured worst-case bound of  $O(\sqrt{\log n})$  is valid. We also address the issue of dimension reduction in  $L_p$  for  $p \in (1, 2)$ . We resolve a question left open in [4] about the impossibility of *linear* dimension reduction in the above cases, and we show that the example of [3, 16] cannot be used to prove a lower bound for the non-linear case. This is accomplished by exhibiting constant-distortion embeddings of snowflaked planar metrics into Euclidean space.

## 1 Introduction

This paper is devoted to the analysis of metric properties of finite subsets of  $L_1$ . Such metrics occur in many important algorithmic contexts, and their analysis is key to progress on some fundamental problems. For instance, an  $O(\log n)$ -approximate max-flow/min-cut theorem proved elusive for many years until, in [18, 2], it was shown to follow from a theorem of Bourgain stating that every metric on  $n$  points embeds into  $L_1$  with distortion  $O(\log n)$ .

The importance of  $L_1$  metrics has given rise to many problems and conjectures that have attracted a lot of attention in recent years. Four basic problems of this type are as follows.

- I. Is there an  $L_1$  analog of the Johnson-Lindenstrauss dimension reduction lemma [12]?
- II. Are all  $n$ -point subsets of  $L_1$   $O(\sqrt{\log n})$ -embeddable into Hilbert space?
- III. Are all squared- $\ell_2$  metrics  $O(1)$ -embeddable into  $L_1$ ?
- IV. Are all planar graphs  $O(1)$ -embeddable into  $L_1$ ?

(We recall that a squared- $\ell_2$  metric is a space  $(X, d)$  for which  $(X, d^{1/2})$  embeds isometrically in a Hilbert space.)

Each of these questions has been asked many times before; we refer to [21, 22, 17, 11], in particular. Despite an immense amount of interest and effort, the metric properties of  $L_1$  have proved quite elusive;

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hence the name “The mysterious  $L_1$ ” appearing in a survey of Linial at the ICM in 2002 [17]. In this paper, we attempt to offer new insights into the above problems and touch on some relationships between them. We refer the reader to the book [21] for an introductory account of the theory of low distortion embeddings of metric spaces. In particular, throughout this paper we shall use the standard terminology appearing in [21].

## 1.1 Results and techniques

**Euclidean distortion.** Our first result addresses problem (II) stated above. We show that the answer to this question is positive on average, in the following sense.

**Theorem 1.1.** *For every  $f_1, \dots, f_n \in L_1$  there is a linear operator  $T : L_1 \rightarrow L_2$  such that*

$$\frac{\|T(f_i) - T(f_j)\|_2}{\|f_i - f_j\|_1} \geq \frac{1}{\sqrt{8 \log n}}, \quad 1 \leq i < j \leq n, \text{ and}$$

$$\frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \left( \frac{\|T(f_i) - T(f_j)\|_2}{\|f_i - f_j\|_1} \right)^{1/2} \leq 10.$$

In other words, for any  $n$ -point subset in  $L_1$ , there exists a map into  $L_2$  such that distances are contracted by at most  $O(\sqrt{\log n})$  and the average expansion is  $O(1)$ . This yields the first positive evidence that the conjectured worst-case bound of  $O(\sqrt{\log n})$  holds. We remark that a different notion of average embedding was recently studied by Rabinovich [24]; there, one tries to embed (planar) metrics into the line such that the *average distance* does not change too much.

The exponent  $1/2$  above has no significance, and we can actually obtain the same result for any power  $1 - \varepsilon$ ,  $\varepsilon > 0$  (we refer to Section 2 for details). The proof of Theorem 1.1 follows from the following probabilistic lemma, which is implicit in [19]. We believe that this result is of independent interest.

**Lemma 1.2.** *There exists a distribution over linear mappings  $T : L_1 \rightarrow L_2$  such that for every  $x \in L_1 \setminus \{0\}$  the random variable  $\frac{\|T(x)\|_2}{\|x\|_1}$  has density  $\frac{e^{-1/(4x^2)}}{x^2 \sqrt{\pi}}$ .*

In contrast to Theorem 1.1, we show that problem (II) cannot be resolved positively using linear mappings. Specifically, we show that there are arbitrarily large  $n$ -point subsets of  $L_1$  such that any linear embedding of them into  $L_2$  incurs distortion  $\Omega(\sqrt{n})$ . As a corollary we settle the problem left open by Charikar and Sahai in [4], whether dimension reduction with a *linear* map is possible in  $L_p$ ,  $p \notin \{1, 2\}$ . The case  $p = 1$  was proved in [4] via linear programming techniques, and it seems impossible to generalize their method to arbitrary  $L_p$ . We show that there are arbitrarily large  $n$ -point subsets  $X \subseteq L_p$  (namely, the same point set used in [4] to handle the case  $p = 1$ ), such that any linear embedding of  $X$  into  $\ell_p^d$  incurs distortion  $\Omega[(n/d)^{|1/p - 1/2|}]$ , thus dimension reduction with a linear map is impossible in any  $L_p$ ,  $p \neq 2$ . Additionally, we show that there are arbitrarily large  $n$ -point subsets  $X \subseteq L_1$  such any linear embedding of  $X$  into *any*  $d$ -dimensional normed space incurs distortion  $\Omega(\sqrt{n/d})$ . This generalizes the Charikar-Sahai result to arbitrary low dimensional norms.

**Dimension reduction.** In [3], and soon after in [16], it was shown that if the Newman-Rabinovich diamond graph on  $n$  vertices  $\alpha$ -embeds into  $\ell_1^d$  then  $d \geq n^{\Omega(1/\alpha^2)}$ . The proof in [3] is based on a linear programming argument, while the proof in [16] uses a geometric argument which reduces the problem to bounding from below the distortion required to embed the diamond graph in  $\ell_p$ ,  $1 < p < 2$ . These results settle the long standing open problem of whether there is an  $L_1$  analog of the Johnson-Lindenstrauss dimension reduction lemma [12]. (In other words, they show that the answer to problem (I) above is *No.*). In Section 4, we show that the method of proof in [16] can be used to provide an even more striking counter example to this problem.

A metric space  $X$  is called *doubling* with constant  $C$  if every ball in  $X$  can be covered by  $C$  balls of half the radius. Doubling metrics with bounded doubling constants are widely viewed as low dimensional (see

[9, 13] for some practical and theoretical applications of this viewpoint). On the other hand, the doubling constant of the diamond graphs is  $\Omega(\sqrt{n})$  (where  $n$  is the number of points). Based on a fractal construction due to Laakso [14] and the method developed in [16], we prove the following theorem, which shows a strong lower bound on the dimension required to represent uniformly doubling subsets of  $L_1$ .

**Theorem 1.3.** *There are arbitrarily large  $n$ -point subsets  $X \subseteq L_1$  which are doubling with constant 6 but such that every  $\alpha$ -embedding of  $X$  into  $\ell_1^d$  requires  $d \geq n^{\Omega(1/\alpha^2)}$ .*

In [15, 9] it was asked whether any subset of  $\ell_2$  which is doubling well-embeds into  $\ell_2^d$  (with bounds on the distortion and the dimension that depend only on the doubling constant). In [9], it was shown that a similar property cannot hold for  $\ell_1$ . Our lower bound exponentially strengthens that result.

**Planar metrics.** Our final result addresses problems (III) and (IV). Our motivation was an attempt to generalize the argument in [16] to prove that dimension reduction is impossible in  $L_p$  for any  $1 < p < 2$ . A natural approach to this problem is to consider the point set used in [3, 16] (namely, a natural realization of the diamond graph,  $G$ , in  $L_1$ ) with the metric induced by the  $L_p$  norm instead of the  $L_1$  norm. This is easily seen to amount to proving lower bounds on the distortion required to embed the metric space  $(G, d_G^{1/p})$  in  $\ell_p^h$ . Unfortunately, this approach cannot work since we show that, for any planar metric  $(X, d)$  and any  $0 < \varepsilon < 1$ , the metric space  $(X, d^{1-\varepsilon})$  embeds in Hilbert space with distortion  $O(1/\sqrt{\varepsilon})$ , and then using results of Johnson and Lindenstrauss [12], and Figiel, Lindenstrauss and Milman [6], we conclude that this metric can be  $O(1/\sqrt{\varepsilon})$  embedded in  $\ell_p^h$ , where  $h = O(\log n)$ . The proof of this interesting fact is a straightforward application of Assouad's classical embedding theorem [1] and Rao's embedding method [25]. The  $O(1/\sqrt{\varepsilon})$  upper bound is shown to be tight for every value  $0 < \varepsilon < 1$ . We note that the case  $\varepsilon = 1/2$  has been previously observed by A. Gupta in his (unpublished) thesis [7].

## 2 Average distortion Euclidean embedding of subsets of $L_1$

The heart of our argument is the following lemma which is implicit in [19], and which seems to be of independent interest.

**Lemma 2.1.** *For every  $0 < p \leq 2$  there is a probability space  $(\Omega, P)$  such that for every  $\omega \in \Omega$  there is a linear operator  $T_\omega : L_p \rightarrow L_2$  such that for every  $x \in L_p \setminus \{0\}$  the random variable  $X = \frac{\|T_\omega(x)\|_2}{\|x\|_p}$  satisfies for every  $a \in \mathbb{R}$ ,  $\mathbb{E}e^{-aX^2} = e^{-a^{p/2}}$ . In particular, for  $p = 1$  the density of  $X$  is  $\frac{e^{-1/(4x^2)}}{x^2\sqrt{\pi}}$ .*

*Proof.* Consider the following three sequences of random variables,  $\{Y_j\}_{j \geq 1}$ ,  $\{\theta_j\}_{j \geq 1}$ ,  $\{g_j\}_{j \geq 1}$ , such that each variable is independent of the others. For each  $j \geq 1$ ,  $Y_j$  is uniformly distributed on  $[0, 1]$ ,  $g_j$  is a standard Gaussian and  $\theta_j$  is an exponential random variable, i.e. for  $\lambda \geq 0$ ,  $P(\theta_j > \lambda) = e^{-\lambda}$ . Set  $\Gamma_j = \theta_1 + \dots + \theta_j$ . By Proposition 1.5. in [19], there is a constant  $C = C(p)$  such that if we define for  $f \in L_p$

$$V(f) = C \sum_{j \geq 1} \frac{g_j}{\Gamma_j^{1/p}} f(Y_j),$$

then  $\mathbb{E}e^{iV(f)} = e^{-\|f\|_p^p}$ .

Assume that the random variables  $\{Y_j\}_{j \geq 1}$  and  $\{\Gamma_j\}_{j \geq 1}$  are defined on a probability space  $(\Omega, P)$  and that  $\{g_j\}_{j \geq 1}$  are defined on a probability space  $(\Omega', P')$ , in which case we use the notation  $V(f) = V(f; \omega, \omega')$ . Define for  $\omega \in \Omega$  a linear operator  $T_\omega : L_p \rightarrow L_2(\Omega', P')$  by  $T_\omega(f) = V(f; \omega, \cdot)$ . Since for every fixed  $\omega \in \Omega$  the random variable  $V(f; \omega, \cdot)$  is Gaussian with variance  $\|T_\omega(f)\|_2^2$ , for every  $a \in \mathbb{R}$ ,  $\mathbb{E}_{P'} e^{iaV(f; \omega, \cdot)} = e^{-a^2 \|T_\omega(f)\|_2^2}$ . Taking expectation with respect to  $P$  we find that,  $\mathbb{E}_P e^{-a^2 \|T_\omega(f)\|_2^2} = e^{-a^p \|f\|_p^p}$ . This implies the required identity. The explicit distribution in the case  $p = 1$  follows from the fact that the inverse Laplace transform of  $x \mapsto e^{-\sqrt{x}}$  is  $y \mapsto \frac{e^{-1/(4y)}}{2\sqrt{\pi y^3}}$  (see for example [26, 5]).  $\square$

*Proof of Theorem 1.1.* Using the notation of lemma 2.1 (in the case  $p = 1$ ) we find that for every  $a > 0$ ,  $\mathbb{E}e^{-aX^2} = e^{-\sqrt{a}}$ . Hence, for every  $a, \varepsilon > 0$  and every  $1 < i < j \leq n$ ,

$$P\left(\frac{\|T_\omega(f_i) - T_\omega(f_j)\|_2}{\|f_i - f_j\|_1} \leq \varepsilon\right) = P\left(e^{-aX^2} \geq e^{-a\varepsilon^2}\right) \leq e^{a\varepsilon^2 - \sqrt{a}}.$$

Choosing  $a = \frac{1}{4\varepsilon^4}$  the above upper bound becomes  $e^{-1/(4\varepsilon^2)}$ . Consider the set

$$A = \bigcap_{1 \leq i < j \leq n} \left\{ \frac{\|T_\omega(f_i) - T_\omega(f_j)\|_2}{\|f_i - f_j\|_1} \geq \frac{1}{\sqrt{8 \log n}} \right\} \subseteq \Omega.$$

By the union bound,  $P(A) > \frac{1}{2}$ , so that

$$\frac{1}{P(A)} \mathbb{E} \left[ \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \left( \frac{\|T_\omega(f_i) - T_\omega(f_j)\|_2}{\|f_i - f_j\|_1} \right)^{1/2} \right] \leq 2\mathbb{E}X^{1/2} = \frac{2}{\sqrt{\pi}} \int_0^\infty x^{1/2} \cdot \frac{e^{-1/(4x^2)}}{x^2} dx < 10.$$

It follows that there exists  $\omega \in A$  for which the operator  $T = T_\omega$  has the desired properties.  $\square$

**Remark 2.2.** There is nothing special about the choice of the power  $1/2$  in Theorem 1.1. When  $p = 1$ ,  $\mathbb{E}X = \infty$  but  $\mathbb{E}X^{1-\varepsilon} < \infty$  for every  $0 < \varepsilon < 1$ , so we may write the above average with the power  $1 - \varepsilon$  replacing the exponent  $1/2$ . Obvious generalizations of Theorem 1.1 hold true for every  $1 < p < 2$ , in which case the average distortion is of order  $C(p)(\log n)^{1/p-1/2}$  (and the power can be taken to be 1).

### 3 The impossibility of dimension reduction with a *linear* map in $L_p$ , $p \neq 2$

The above method cannot yield a  $O(\sqrt{\log n})$  bound on the Euclidean distortion of  $n$ -point subsets of  $L_1$ . In fact, there are arbitrarily large  $n$ -point subsets of  $L_1$  on which any *linear* embedding into  $L_2$  incurs distortion at least  $\sqrt{\frac{n-1}{2}}$ . This follows from the following simple lemma:

**Lemma 3.1.** *For every  $1 \leq p \leq \infty$  there are arbitrarily large  $n$ -point subsets of  $L_p$  on which any linear embedding into  $L_2$  incurs distortion at least  $\left(\frac{n-1}{2}\right)^{|1/p-1/2|}$ .*

*Proof.* Let  $w_1, \dots, w_{2^k}$  be the rows of the  $2^k \times 2^k$  Walsh matrix (i.e. the simplest Hadamard matrix). Write  $w_i = \sum_{j=1}^{2^k} w_{ij} e_j$  where  $e_1, \dots, e_{2^k}$  are the standard unit vectors in  $\mathbb{R}^{2^k}$ . Consider the set  $A = \{0\} \cup \{w_i\}_{i=1}^{2^k} \cup \{e_i\}_{i=1}^{2^k} \subset \ell_p$ . Let  $T : \ell_p \rightarrow L_2$  be any linear operator which is non contracting and  $L$ -Lipschitz on  $A$ . Assume first of all that  $1 \leq p < 2$ . Then:

$$\begin{aligned} 2^{k(1+2/p)} &= \sum_{i=1}^{2^k} \|w_i\|_p^2 \leq \sum_{i=1}^{2^k} \|Tw_i\|_2^2 = \sum_{i=1}^{2^k} \left\| \sum_{j=1}^{2^k} w_{ij} T(e_j) \right\|_2^2 \\ &= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} \langle w_i, w_j \rangle \langle T(e_i), T(e_j) \rangle = 2^k \sum_{j=1}^{2^k} \|T(e_j)\|_2^2 \leq 4^k \cdot L^2, \end{aligned}$$

which implies that  $L \geq 2^{k(1/p-1/2)} = \left(\frac{|A|-1}{2}\right)^{1/p-1/2}$ . When  $p > 2$  apply the same reasoning, with the inequalities reversed.  $\square$

We remark that the above point set was also used by Charikar and Sahai [4] to give a lower bound on dimension reduction with a linear map in  $L_1$ . Their proof used a linear programming argument, which doesn't seem to be generalizable to the case of  $L_p$ ,  $p > 1$ . Lemma 3.1 formally implies their result (with a significantly simpler proof), and in fact proves the impossibility of dimension reduction with a linear map in any  $L_p$ ,  $p \neq 2$ . Indeed, if there were a linear operator which embeds  $A$  into  $\ell_p^d$  with distortion  $D$  then it would also be a  $D \cdot d^{|1/p-1/2|}$  embedding into  $\ell_2^d$ . It follows that  $D \geq \left(\frac{|A|-1}{2d}\right)^{|1/p-1/2|}$ . Similarly, since by John's theorem (see e.g. [23]) any  $d$ -dimensional normed space is  $\sqrt{d}$  equivalent to Hilbert space, we deduce that there are arbitrarily large  $n$ -point subsets of  $L_1$ , any linear embedding of which into any  $d$ -dimensional normed space incurs distortion at least  $\sqrt{\frac{n-1}{2d}}$ .

## 4 An inherently high-dimensional doubling metric in $L_1$

This section is devoted to the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Consider the Laakso graphs,  $\{G_i\}_{i=0}^\infty$ , which are defined as follows.  $G_0$  is the graph on two vertices with one edge. To construct  $G_i$ , take six copies of  $G_{i-1}$  and scale their metric by a factor of  $\frac{1}{4}$ . We glue four of them cyclicly by identifying pairs of endpoints, and attach at two opposite gluing points the remaining two copies. See Figure 1 below.

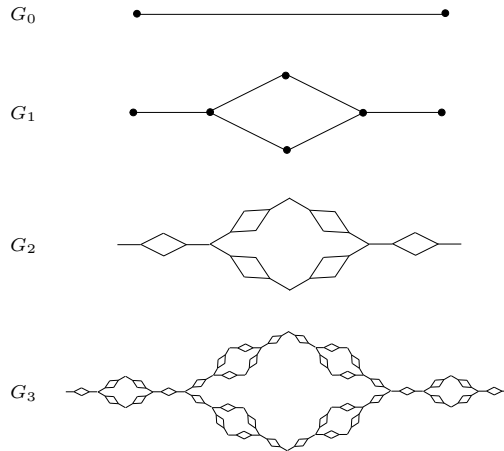


Figure 1: The Laakso graphs.

As shown in [14], the graphs  $\{G_i\}_{i=0}^\infty$  are uniformly doubling (see also [15], for a simple argument showing they are doubling with constant 6). Moreover, since the  $G_i$ 's are series parallel graphs, they embed uniformly in  $L_1$  (see [8]).

We will show below that any embedding of  $G_i$  in  $L_p$ ,  $1 < p \leq 2$  incurs distortion at least  $\sqrt{1 + \frac{p-1}{4}i}$ . We then conclude as in [16] by observing that  $\ell_1^d$  is 3-isomorphic to  $\ell_p^d$  when  $p = 1 + \frac{1}{\log d}$ , so that if  $G_i$  embeds with distortion  $\alpha$  in  $\ell_1^d$  then  $\alpha \geq \sqrt{\frac{i}{40 \log d}}$ . This implies the required result since  $i \approx \log |G_i|$ .

The proof of the lower bound for the distortion required to embed  $G_i$  into  $L_p$  is by induction on  $i$ . We shall prove by induction that whenever  $f : G_i \rightarrow L_p$  is non-contracting then there exist two adjacent vertices  $u, v \in G_i$  such that  $\|f(u) - f(v)\|_p \geq d_{G_i}(u, v) \sqrt{1 + \frac{p-1}{4}i}$  (observe that for  $u, v \in G_{i-1}$ ,  $d_{G_{i-1}}(u, v) = d_{G_i}(u, v)$ ). For  $i = 0$  there is nothing to prove. For  $i \geq 1$ , since  $G_i$  contains an isometric copy of  $G_{i-1}$ , there are  $u, v \in G_i$  corresponding to two adjacent vertices in  $G_{i-1}$  such that  $\|f(u) - f(v)\|_p \geq d_{G_i}(u, v) \sqrt{1 + \frac{p-1}{4}(i-1)}$ . Let

$a, b$  be the two midpoints between  $u$  and  $v$  in  $G_i$ . By Lemma 2.1 in [16],

$$\begin{aligned} \|f(u) - f(v)\|_p^2 + (p-1)\|f(a) - f(b)\|_p^2 \\ \leq \|f(u) - f(a)\|_p^2 + \|f(a) - f(v)\|_p^2 + \|f(v) - f(b)\|_p^2 + \|f(b) - f(u)\|_p^2. \end{aligned}$$

Hence:

$$\begin{aligned} & \max\{\|f(u) - f(a)\|_p^2, \|f(a) - f(v)\|_p^2, \|f(v) - f(b)\|_p^2, \|f(b) - f(u)\|_p^2\} \\ & \geq \frac{1}{4}\|f(u) - f(v)\|_p^2 + \frac{1}{4}(p-1)\|f(a) - f(b)\|_p^2 \\ & \geq \frac{1}{4}\left(1 + \frac{p-1}{4}(i-1)\right)d_{G_i}(u, v)^2 + \frac{p-1}{4}d_{G_i}(a, b)^2 \\ & = \frac{1}{4}\left(1 + \frac{p-1}{4}i\right)d_{G_i}(u, v)^2 \\ & = \left(1 + \frac{p-1}{4}i\right)\max\{d_{G_i}(u, a)^2, d_{G_i}(a, v)^2, d_{G_i}(v, b)^2, d_{G_i}(b, u)^2\}. \end{aligned}$$

□

We end this section by observing that the above approach also gives a lower bound on the dimension required to embed expanders in  $\ell_\infty$ .

**Proposition 4.1.** *Let  $G$  be an  $n$ -point constant degree expander which embeds in  $\ell_\infty^d$  with distortion at most  $\alpha$ . Then  $d \geq n^{\Omega(1/\alpha)}$ .*

*Proof.* By Matoušek's lower bound for the distortion required to embed expanders in  $\ell_p$  [20], any embedding of  $G$  into  $\ell_p$  incurs distortion  $\Omega\left(\frac{\log n}{p}\right)$ . Since  $\ell_\infty^d$  is  $O(1)$ -equivalent to  $\ell_{\log d}^d$ , we deduce that  $\alpha \geq \Omega\left(\frac{\log n}{\log d}\right)$ . □

We can also obtain a lower bound on the dimension required to embed the Hamming cube  $\{0, 1\}^k$  into  $\ell_\infty$ . Our proof uses a simple concentration argument. An analogous concentration argument yields an alternative proof of Proposition 4.1.

**Proposition 4.2.** *Assume that  $\{0, 1\}^k$  embeds into  $\ell_\infty^d$  with distortion  $\alpha$ . Then  $d \geq 2^{k\Omega(1/\alpha^2)}$ .*

*Proof.* Let  $f = (f_1, \dots, f_d) : \{0, 1\}^k \rightarrow \ell_\infty^d$  be a contraction such that for every  $u, v \in \{0, 1\}^d$ ,  $\|f(u) - f(v)\|_\infty \geq \frac{1}{\alpha}d(u, v)$  (where  $d(\cdot, \cdot)$  denotes the Hamming metric). Denote by  $P$  the uniform probability measure on  $\{0, 1\}^k$ . Since for every  $1 \leq i \leq k$ ,  $f_i$  is 1-Lipschitz, the standard concentration inequality on the hypercube (see [21]) implies that  $P(|f_i(u) - \mathbb{E}f_i| \geq k/(4\alpha)) \leq 2e^{-k/(32\alpha^2)}$ . On the other hand, if  $u, v \in \{0, 1\}^k$  are such that  $d(u, v) = k$  then there exist  $1 \leq i \leq d$  for which  $|f_i(u) - f_i(v)| \geq k/\alpha$ , implying that  $\max\{|f_i(u) - \mathbb{E}f_i|, |f_i(v) - \mathbb{E}f_i|\} > k/(4\alpha)$ . By the union bound it follows that  $de^{-\Omega(k/\alpha^2)} \geq 1$ , as required. □

## 5 Snowflake versions of planar metrics

The problem of whether there is an analog of the Johnson-Lindenstrauss dimension reduction lemma in  $L_p$ ,  $1 < p < 2$ , is an interesting one which remains open. In view of the above proof and the proof in [16], a natural point set which is a candidate to demonstrate the impossibility of dimension reduction in  $L_p$  is the realization of the diamond graph in  $\ell_1$  which appears in [3], equipped with the  $\ell_p$  metric. Since this point set consists of vectors whose coordinates are either 0 or 1 (i.e. subsets of the cube), this amounts to considering the diamond graph with its metric raised to the power  $\frac{1}{p}$ . Unfortunately, this approach cannot work; we show below that any planar graph whose metric is raised to the power  $1 - \varepsilon$  has Euclidean distortion  $O(1/\sqrt{\varepsilon})$ .

Given a metric space  $(X, d)$  and  $\varepsilon > 0$ , the metric space  $(X, d^{1-\varepsilon})$  is known in geometric analysis (see e.g. [10]) as the  $1 - \varepsilon$  snowflake version of  $(X, d)$ . Assouad's classical theorem [1] states that any snowflake version of a doubling metric space is bi-Lipschitz equivalent to a subset of some finite dimensional Euclidean space. A quantitative version of this result (with bounds on the distortion and the dimension) was obtained in [9]. The following theorem is proved by combining embedding techniques of Rao [25] and Assouad [1]. A similar analysis is also used in [9]. In what follows we call a metric  $K_r$ -excluded if it is the metric on a subset of a weighted graph which does not admit a  $K_r$  minor. In particular, planar metrics are all  $K_5$ -excluded.

**Theorem 5.1.** *For any  $r \in \mathbb{N}$  there exists a constant  $C(r)$  such that for every  $0 < \varepsilon < 1$ , a  $1 - \varepsilon$  snowflake version of a  $K_r$ -excluded metric embeds into  $\ell_2$  with distortion at most  $C(r)/\sqrt{\varepsilon}$ .*

Our argument is based on the following lemma, the proof of which is contained in [25].

**Lemma 5.2.** *For every  $r \in \mathbb{N}$  there is a constant  $\delta = \delta(r)$  such that for every  $\rho > 0$  and every  $K_r$ -excluded metric  $(X, d)$  there exists a finitely supported probability distribution  $\mu$  on partitions of  $X$  with the following properties:*

1. *For every  $P \in \text{supp}(\mu)$ , and for every  $C \in P$ ,  $\text{diam}(C) \leq \rho$ .*
2. *For every  $x \in X$ ,  $\mathbb{E}_\mu \sum_{C \in P} d(x, X \setminus C) \geq \delta \rho$ .*

Observe that the sum under the expectation in (2) above actually consists of only one summand.

*Proof of Theorem 5.1.* Let  $X$  be a  $K_r$ -excluded metric. For each  $n \in \mathbb{Z}$ , we define a map  $\phi_n$  as follows. Let  $\mu_n$  be the probability distribution on partitions of  $X$  from Lemma 5.2 with  $\rho = 2^{n/(1-\varepsilon)}$ . Fix a partition  $P \in \text{supp}(\mu_n)$ . For any  $\sigma \in \{-1, +1\}^{|P|}$ , consider  $\sigma$  to be indexed by  $C \in P$  so that  $\sigma_C$  denotes the value of  $\sigma$  at  $C$ . Following Rao [25], define

$$\phi_P(x) = \bigoplus_{\sigma \in \{-1, +1\}^{|P|}} \sqrt{\frac{1}{2^{|P|}}} \sum_{C \in P} \sigma_C \cdot d(x, X \setminus C),$$

and write  $\phi_n = \bigoplus_{P \in \text{supp}(\mu_n)} \sqrt{\mu_n(P)} \phi_P$  (here the symbol  $\bigoplus$  refers to the concatenation operator).

Now, following Assouad [1], let  $\{e_i\}_{i \in \mathbb{Z}}$  be an orthonormal basis of  $\ell_2$ , and set

$$\Phi(x) = \sum_{n \in \mathbb{Z}} 2^{-n\varepsilon/(1-\varepsilon)} \phi_n(x) \otimes e_n$$

**Claim 5.3.** *For every  $n \in \mathbb{Z}$ , and  $x, y \in X$ , we have  $\|\phi_n(x) - \phi_n(y)\|_2 \leq 2 \cdot \min\{d(x, y), 2^{n/(1-\varepsilon)}\}$ . Additionally, if  $d(x, y) > 2^{n/(1-\varepsilon)}$ , then  $\|\phi_n(x) - \phi_n(y)\|_2 \geq \delta 2^{n/(1-\varepsilon)}$ .*

*Proof.* For any partition  $P \in \text{supp}(\mu_n)$ , let  $C_x, C_y$  be the clusters of  $P$  containing  $x$  and  $y$ , respectively. Note that since for every  $C \in P$ ,  $\text{diam}(C) \leq 2^{n/(1-\varepsilon)}$ , when  $d(x, y) > 2^{n/(1-\varepsilon)}$ , we have  $C_x \neq C_y$ . In this case,

$$\begin{aligned} \|\phi_P(x) - \phi_P(y)\|_2^2 &= \mathbb{E}_{\sigma \in \{-1, +1\}^{|P|}} |\sigma_{C_x} d(x, X \setminus C_x) - \sigma_{C_y} d(y, X \setminus C_y)|^2 \\ &\geq \frac{d(x, X \setminus C_x)^2 + d(y, X \setminus C_y)^2}{2}. \end{aligned}$$

It follows that

$$\begin{aligned} \|\phi_n(x) - \phi_n(y)\|_2^2 &= \mathbb{E}_{\mu_n} \|\phi_P(x) - \phi_P(y)\|_2^2 \\ &\geq \frac{\mathbb{E}_{\mu_n} d(x, X \setminus C_x)^2 + \mathbb{E}_{\mu_n} d(y, X \setminus C_y)^2}{2} \geq \left(\delta 2^{n/(1-\varepsilon)}\right)^2. \end{aligned}$$

On the other hand, for every  $x, y \in X$ , since  $d(x, X \setminus C_x), d(y, X \setminus C_y) \leq 2^{n/(1-\varepsilon)}$ , we have that  $\|\phi_P(x) - \phi_P(y)\|_2 \leq 2 \cdot \min\{d(x, y), 2^{n/(1-\varepsilon)}\}$ , hence  $\|\phi_n(x) - \phi_n(y)\|_2 \leq 2 \cdot \min\{d(x, y), 2^{n/(1-\varepsilon)}\}$ .  $\square$



To finish the analysis, let us fix  $x, y \in X$  and let  $m$  be such that  $d(x, y)^{1-\varepsilon} \in (2^m, 2^{m+1}]$ . In this case,

$$\begin{aligned}
\|\Phi(x) - \Phi(y)\|_2^2 &= \sum_{n \in \mathbb{Z}} 2^{-2n\varepsilon/(1-\varepsilon)} \|\phi_n(x) - \phi_n(y)\|_2^2 \\
&\leq 4 \sum_{n < m} 2^{2n} + 4d(x, y)^2 \sum_{n \geq m} 2^{-2n\varepsilon/(1-\varepsilon)} \\
&= 2^{2m+1} + 4d(x, y)^2 \frac{2^{-2m\varepsilon/(1-\varepsilon)}}{1 - 2^{-2\varepsilon/(1-\varepsilon)}} \\
&= O(1/\varepsilon) \cdot d(x, y)^{2(1-\varepsilon)}
\end{aligned}$$

On the other hand,

$$\|\Phi(x) - \Phi(y)\|_2 \geq 2^{-m\varepsilon/(1-\varepsilon)} \|\phi_m(x) - \phi_m(y)\|_2 \geq \delta 2^m \geq \frac{\delta}{2} d(x, y)^{1-\varepsilon}.$$

The proof is complete.  $\square$

**Remark 5.4.** The  $O(1/\sqrt{\varepsilon})$  upper bound in Theorem 5.1 is tight. In fact, for  $i \approx 1/\varepsilon$ , the  $1 - \varepsilon$  snowflake version of the Laakso graph  $G_i$  (presented in Section 4) has Euclidean distortion  $\Omega(1/\sqrt{\varepsilon})$ . To see this, let  $f : G_i \rightarrow \ell_2$  be any non-contracting embedding of  $(G_i, d_{G_i}^{1-\varepsilon})$  into  $\ell_2$ . For  $j \leq i$  denote by  $K_j$  the Lipschitz constant of the restriction of  $f$  to  $(G_j, d_{G_i}^{1-\varepsilon})$  (as before, we think of  $G_j$  as a subset of  $G_i$ ). Clearly  $K_0 = 1$ , and the same reasoning as in the proof of Theorem 1.3 shows that for  $j \geq 1$ ,  $K_j^2 \geq \frac{K_{j-1}^2}{4^\varepsilon} + \frac{1}{4}$ . This implies that  $K_i^2 \geq \frac{1}{4} + \frac{1}{4^\varepsilon} + \dots + \frac{1}{4^{i\varepsilon}} = \Omega(1/\varepsilon)$ , as required.

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